Primary decomposable subspaces of $k[t]$ and Right ideals of the first Weyl algebra $A_1(k)$ in characteristic zero

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In the classification of right ideals $A_1 := k[t, \partial]$ the first Weyl algebra over a field k , R. Cannings and M.P. Holland established in [3, Theorem 0.5] a bijective correspondence between primary decomposable subspaces of $R = k[t]$ and right ideals I of $A_1 := k[t, \partial]$ the first Weyl algebra over k which have non-trivial intersection with $k[t]$:

$$\Gamma : V \longmapsto \mathcal{D}(R, V) \ \ , \ \ \Gamma^{-1} : I \longmapsto I \star 1$$

This theorem is a very important step in this study, after Stafford's theorem [1, Lemma 4.2]. However, the theorem had been established only when the field k is an algebraically closed and of characteristic zero.

In this paper we define notion of primary decomposable subspaces of $k[t]$ when k is any field of characteristic zero, particulary for $\mathbb{Q}$, $\mathbb{R}$, and we show that R. Cannings and M.P. Holland's correspondence theorem holds. Thus right ideals of $A_1(\mathbb{Q})$, $A_1(\mathbb{R})$,.. are also described by this theorem.

1 Cannings and Holland's theorem

1.1 Weyl algebra in characteristic zero and differential operators

Let k be a commutative field of characteristic zero and $A_1 := A_1(k) = k[t, \partial]$ where ∂, t are related by $\partial t - t\partial = 1$, be the first Weyl algebra over k.

A_1 contains the subring $R := k[t]$ and $S := k[\partial]$. It is well known that A_1 is an integral domain, two-sided noetherian and since the characteristic of k is zero, A_1 is hereditary (see [2]). In particular, A_1 has a quotient divison ring, denoted by Q_1. For any right (resp: left) submodule of Q_1, M^* the dual as A_1-module will be identified with the set $\{u \in Q_1 : uM \subset A_1\}$ (resp:$\{u \in Q_1 : Mu \subset A_1\}$) when M is finitely generated (see [1]).

Q_1 contains the subrings $D = k(t)[\partial]$ and $B = k(\partial)[t]$. The elements of D are k-linear endomorphisms of $k(t)$. Precisely, if $d = a_n\partial^n + \cdots + a_1\partial + a_0$ where $a_i \in k(t)$ and $h \in k(t)$, then

$$d(h) := a_n h^{(n)} + \cdots + a_1 h^{(1)} + a_0 h$$

where $h^{(i)}$ denotes the i-th derivative of h and $a_i h^{(i)}$ is a product in $k(t)$. One checks that:

$$(dd')(h) = d(d'(h)) \textbf{ for } d, d' \in k(t)[\partial] \ , \ h \in k(t)$$

For V and W two vector subspaces of $k(t)$, we set :

$$\mathcal{D}(V, W) := \{d \in k(t)[\partial] : d(V) \subset W\}$$

$\mathcal{D}(V, W)$ is called the set of differential operators from V to W.

Notice that $\mathcal{D}(R, V)$ is an A_1 right submodule of Q_1 and $\mathcal{D}(V, R)$ is an A_1 left submodule of Q_1. If $V \subseteq R$, one notes that $\mathcal{D}(R, V)$ is a right ideal of A_1. When $V = R$, then $\mathcal{D}(R, R) = A_1$.

If I is a right ideal of A_1, we set

$$I \star 1 := \{d(1), d \in I\}$$

Clearly, $I \star 1$ is a vector subspace of $k[t]$ and $I \subseteq \mathcal{D}(R, I \star 1)$.

Inclusion $A_1 \subset k(\partial)[t]$ and $A_1 \subset k(t)[\partial]$ show that it can be defined on A_1 two notions of degree: the degree associated to "t" and the degree associated to "∂". Naturally, those degree notions extend to Q_1.

1.2 Stafford's theorem

Let I be a non-zero right ideal of A_1. By J. T. Stafford in [1, Lemma 4.2], there exist $x, e \in Q_1$ such that:

$$(i)\ xI \subset A_1 \ \texttt{and}\ \ xI \cap k[t] \neq \{0\} \ \ \textbf{,}\ \ (ii)\ eI \subset A_1 \ \texttt{and}\ \ eI \cap k[\partial] \neq \{0\}$$

By (i) one sees that any non-zero right ideal I of A_1 is isomorphic to another ideal I' such that $I' \cap k[t] \neq \{0\}$, which means that I' has non-trivial intersection with $k[t]$. We denote $\mathcal{I}_t$ the set of right ideals I of A_1 the first Weyl algebra over k such that $I \cap k[t] \neq \{0\}$

Stafford's theorem is the first step in the classification of right ideals of the first Weyl algebra A_1.

1.3 The bijective correspondence theorem

Let c be an algebraically closed field of characteristic zero. Cannings and Holland have defined primary decomposable subspace V of $c[t]$ as finite intersections of primary subspaces which are vector subspaces of $c[t]$ containing a power of a maximal ideal m of $c[t]$. Since c is an algebraically closed field, maximal ideals of $c[t]$ are generated by one polynomial of degree one: $m = (t-\lambda)c[t]$. So, a vector subspace V of $c[t]$ is primary decomposable if:

$$V = \bigcap_{i=1}^{n} V_i$$

where each V_i contains a power of a maximal ideal m_i of $c[t]$.

They have established the nice well-known bijective correspondence between primary decomposable subspaces of $c[t]$ and $\mathcal{I}_t$ by:

$$\Gamma : V \longmapsto \mathcal{D}(R, V) \ \ \textbf{,}\ \ \Gamma^{-1} : I \longmapsto I \star 1$$

Since $V = \bigcap_{i=1}^{n} V_i$ and $m_i =< (t-\lambda_i)^{r_i} > \subseteq V_i$,

one has $(t-\lambda_1)^{r_1} \cdots (t-\lambda_n)^{r_n} k[t] \subseteq V$. So, easily one sees that

$$(t-\lambda_1)^{r_1} \cdots (t-\lambda_n)^{r_n} k[t] \subseteq \mathcal{D}(R, V) \cap c[t]$$

However it is not clear that $I \star 1$ must be a primary decomposable subspace of $c[t]$.

Cannings and Holland's theorem use the following result, which holds even if the field is just of characteristic zero:

Lemma 1: Let $I \in \mathcal{I}_t$ and $V = I \star 1$. One has:
$I = \mathcal{D}(R, V)$ and $I^* = \mathcal{D}(V, R)$.

For the proof of Cannings and Holland's theorem one can see [3].

We note that, since $< (t-\lambda_i)^{r_i} > \subseteq V_i$, for any s in the ring $c + (t-\lambda_i)^{r_i} c[t]$, one has :

$$s \cdot V_i \subseteq V_i$$

It is this remark which will allow us to give general definition of primary decomposable subspaces of $k[t]$ for any field k of characteristic zero, not necessarily algebraically closed.

2 Primary decomposable subspaces of $k[t]$

Here we give a general definition of primary decomposable subspaces of $k[t]$ when k is any field of characteristic zero not necessarily algebraically closed and we keep the bijective correspondence of Cannings and Holland.

2.1 Definitions and examples

•- Definitions

Let $b, h \in R = k[t]$ and V a k-subspace of $k[t]$. We set:

$$O(b) = \{a \in R : a' \in bR\} \text{ and } O(b, h) = \{a \in R : a' + ah \in bR\}$$

where a' denotes the formal derivative of a.

$$S(V) = \{a \in R : aV \subseteq V\} \text{ and } C(R, V) = \{a \in R : aR \subseteq V\}$$

Clearly $O(b)$ and $S(V)$ are k-subalgebras of $k[t]$. If $b \neq 0$, the Krull dimension of $O(b)$ is $dim_K(O(b)) = 1$. The set $C(R, V)$ is an ideal of R contained in both $S(V)$ and V .

• A k-vector subspace V of $k[t]$ is said to be primary decomposable if $S(V)$ contains a k-subalgebra $O(b)$, with $b \neq 0$.

•- Examples

◦ Easily one sees that $O(b) \subseteq S(O(b,h))$ and $C(R, O(b,h)) = C(R, O(b))$, in particular $O(b,h)$ is a primary decomposable subspace when $b \neq 0$.

Following lemmas and corollary show that classical primary decomposable subspaces are primary decomposable in the new way.

Lemma 2: Let k be a field of characteristic zero and λ_1, ..,λ_n finite distinct elements of k. Suppose that V_1 , ..,V_n are k-vector subspaces $k[t]$, each V_i contains $(t-\lambda_i)^{r_i}k[t]$ for some $r_i \in \mathbb{N}^*$. Then

$$O((t-\lambda_1)^{r_1-1} \cdots (t-\lambda_n)^{r_n-1}) \subseteq S(\bigcap_{i=1}^{n} V_i)$$

Proof: : One has $O((t-\lambda_i)^{r_i-1}) = k + (t-\lambda_i)^{r_i}k[t]$ and

$$O((t-\lambda_i)^{r_i-1} \cdot\cdot (t-\lambda_n)^{r_n-1}) = \bigcap_{i=1}^{n} O((t-\lambda_i)^{r_i-1})$$

An immediate consequence of this lemma is:

Corollary 3: In the above hypothesis of lemma 2, let

$$V = \bigcap_{i=1}^{n} V_i$$

. If $q \in C(R,V)$, then $O(q) \subseteq S(V)$.

Proof:: First one notes that if $q \in pk[t]$, then $O(q) \subseteq O(p)$. Let $b = (t-\lambda_1)^{r_1} \cdot\cdot (t-\lambda_n)^{r_n}$.

In the above hypothesis, one has

$$C(R,V) = \bigcap_{i=1}^{n} C(R,V_i) = \bigcap_{i=1}^{n} (t-\lambda_i)^{r_i}k[t] = (\prod_{i=1}^{n}(t-\lambda_i)^{r_i})k[t] = bk[t]$$

Since $b \in (t-\lambda_1)^{r_1-1} \cdot\cdot (t-\lambda_n)^{r_n-1}k[t] = b_0k[t]$, one has $O(b_0)V_i \subseteq V_i$ for all i, so

$$O(b_0) \subseteq S(V) \texttt{ and } O(q) \subseteq O(b) \subseteq O(b_0)$$

◦ An opposite-example:

Suppose the field k is of characteristic zero and one can find $q \in k[t]$ such that: q is irreducible and $deg(q) \geq 2$. Then the vector subspace $V = k + qk[t]$ is not primary decomposable.

2.2 Classical properties of primary decomposable subspaces

Here we prove that when the field k is algebraically closed of characteristic zero, those two definitions are the same.

Lemma 4: Let k be an algebraically closed field of characteristic zero and V be a k-vector subspace of $k[t]$ such that $S(V)$ contains a k-subalgebra $O(b)$ where $b \neq 0$. Then V is a finite intersections of subspaces which contains a power of a maximal ideal of $k[t]$.

Proof: Since k is algebraically closed field and $b \neq 0$, one can suppose $b = (t-\lambda_1)^{r_1} \cdot\cdot (t-\lambda_n)^{r_n}$. Let $b^* = (t-\lambda_1)\cdot\cdot(t-\lambda_n)$. One has

$$O(b) = \bigcap_{i=1}^{n} (k + (t-\lambda_i)^{r_i+1} R)$$

If we suppose that V is not contained in any ideal of R, one has $V.R = R$. Clearly

$$bb^* R = \prod_{i=1}^{n} (t-\lambda_i)^{r_i+1} R \subset O(b)$$

so $bb^* R = (bb^*)(RV) = (bb^* R)V = bb^* R \subset V$ (1). One also has

$$O(b) \cap (t-\lambda_i)R \neq O(b) \cap (t-\lambda_j)R \textbf{ for all } \ i \neq j$$

in particular one has

$$O(b) = [O(b) \cap (t-\lambda_i)R]^{r_i+1} + [O_k(b) \cap (t-\lambda_j)R]^{r_j+1} \quad (2)$$

With (1) and (2) one gets inductively:

$$V = \bigcap_{i=1}^{n} (V + (t-\lambda_i)^{r_i+1} R) \qquad \diamondsuit$$

One also obtains usual properties of primary decomposable subspaces.

Lemma 5: Let k be a field of characteristic zero, V and W be primary decomposable subspaces of $k[t]$

(1) then $V + W$ and $V \cap W$ are primary decomposable subspaces.

(2) If $q \in k(t)$ such that $qV \subseteq k[t]$, then qV is a primary decomposable subspace.

Proof: One notes that $O(ab) \subseteq O(a) \cap O(b)$ for all a, $b \in k[t]$.

Let us recall basic properties on the subspace $O(a, h)$.
Lemma 6 :
(1) $O(a) \subseteq S(O(a, h))$
(2) $C(R, O(a)) = C(R, O(a, h))$
(3) $a^2 k[t] \subset O(a) \cap O(a, h)$
(4) $\mathcal{D}(R, O(a, h)) = A_1 \cap (\partial + h)^{-1} a A_1$
(5) the subspace $O(a, h)$ is not contained in any proper ideal of R.
(6) For all $q \in O(a, h)$ such that $hcf(q, a) = 1$, one has

$$O(a, h) = qO(a) + C(R, O(a))$$

Proof: One obtains (1), (2), (3), (4) by a straightforward calculation.
Suppose $O(a, h) \subseteq gk[t]$. Then $\mathcal{D}(R, O(a, h)) \subseteq gA_1$, and applying the k-automorphism $\sigma \in Aut_k(A_1)$ such that $\sigma(t) = t$ and $\sigma(\partial) = \partial - h$, one obtains $\mathcal{D}(R, O(a)) \subseteq gA_1$. Clearly the element $f = \partial^{-1} a \partial^{m+1}$ where $deg_t(a) = m$ belongs to $\mathcal{D}(R, O(a)) = A_1 \cap \partial^{-1} a A_1$. When one writes f in extension, one gets exactly

$$f = a\partial^m + a_{m-1}\partial^{m-1} + \cdots + a_1\partial + (-1)^m m!$$

Since $f \in gA_1$, $(-1)^m m!$ must belong to gR. Hence $g \in k^\star$ and one gets (5).
Let q be an element of $O(a, h)$ such that $hcf(q, a) = 1$. One has also $hcf(q, a^2) = 1$, and by Bezout theorem there exist u, $v \in k[t]$ such that:

$$uq + va^2 = 1 \quad (*)$$

The inclusion $qO(a) + C(R, O(a)) \subseteq O(a, h)$ is clear since $q \in O(a, h)$ and one has properties (1) and (2). Conversely let $p \in O(a, h)$. Using $(*)$, one gets

$$p = (pu)q + a^2 pv \quad (**)$$

One notes that $p(uq) = p - pva^2 \in O(a, h)$, so $(p(uq))' + (p(uq))h \in aR$. One has $(p(uq))' + (p(uq))h = p'(uq) + p(uq)' + p(uq)h = p(uq)' + uq(p' + ph)$.

Since q is chosen in $O(a,h)$, one has $p' + ph \in aR$. Then $q(up)' \in aR$, and at the end, because of $hcf(q,a) = 1$, it follows that $(up)' \in aR$. Now, $up \in O(a)$ and $(**)$ shows that $p \in qO(a) + C(R, O(a))$.

Proposition 7: Let k be a field of characteristic zero and V a k-vector subspace of $k[t]$ such that $S(V)$ contains a k-subalgebra $O(b)$. Then

$$\mathcal{D}(R,V) \star 1 = V$$

Proof :
• Suppose $V = O(b)$. One has $\mathcal{D}(R, O(b)) = A_1 \cap \partial^{-1} b A_1$.
Suppose $b = \beta_0 + \beta_1 t + \cdots + \beta_m t^m$ **,** $\beta_m \neq 0$. Then $f = \partial^{-1} b \partial^{m+1} \in A_1 \cap \partial^{-1} b A_1$. Let us show that $f(R) = O(b)$. For an integer $0 \leq p \leq m$, one has:

$$\partial^{-1} t^p \partial^{m+1} = (t\partial - 1) \cdot (t\partial - 2) \cdots (t\partial - p) \partial^{m-p}$$

and so

$$f = \beta_0 \partial^m + \sum_{p=1}^{m} \beta_p (t\partial - 1) \cdot (t\partial - 2) \cdots (t\partial - p) \partial^{m-p}$$

In particular one sees that:
(1) $f(1) = \beta_m (-1)^m m! \neq 0$
(2) $f(t^j) = 0$ **if** $1 \leq j < m$
(3) $f(t^m) = \beta_0 m!$
(4) $deg(f(t^j)) = j$ **when** $j \geq m+1$
It follows that

$$dim \frac{R}{f(R)} = m = dim \frac{R}{O(b)}$$

and since $f(R) \subseteq O(b)$, one gets $f(R) = O(b)$
•• Suppose that $O(b) \subseteq S(V)$. One has $VO(b) = V$ and then

$$[V\mathcal{D}(R, O(b))] \star 1 = V[\mathcal{D}(R, O(b)) \star 1] = VO(b) = V$$

By lemma 1 the equality $V\mathcal{D}(R, O(b)) = \mathcal{D}(R,V)$ holds, so
$\mathcal{D}(R,V) \star 1 = V$.

Next theorem is the main result of this paper.

Theorem 8: Let k be a field of characteristic zero and V a k-vector subspace of $k[t]$ such that: $C(R,V) = qk[t]$ with $q \neq 0$ and $\mathcal{D}(R,V) \star 1 = V$. Then $S(V)$ contains some k-subalgebra $O(b)$ with $b \neq 0$.

Proof: One has $qk[t] \subseteq V$, and there exist v_0,v_1,...,v_m in V such that

$$V =< v_0, v_1, \cdot\cdot, v_m > \oplus qk[t]$$

where $< v_0, v_1, \cdot\cdot, v_m >$ denotes the vector subspace of V generate by $\{v_0, v_1, \cdot\cdot, v_m\}$. For each v_i, there exist $f_i \in \mathcal{D}(R, V)$ such that $f_i(1) = v_i$. Let $r = max\{deg_\partial(f_i),\ 0 \leq i \leq m\}$, we prove that $O(q^r) \cdot V \subseteq V$.

Since the ideal $qk[t]$ of $R = k[t]$ is contained in V, we have only to prove that:

$$O(q^r) \cdot v_i \subseteq V \quad \forall 0 \leq i \leq m$$

We need the following lemma

Lemma 9: Let $d = a_p\partial^p + \cdot\cdot +a_1\partial + a_0 \in A_1(k)$ where $p \in \mathbb{N}$, $b \in k[t]$ and $s \in O(b^p)$. Then $[d, s] = d \cdot s - s \cdot d \in bA_1$.

Proof: One has $[d, s] = [d_1\partial, s] = [d_1, s]\partial + d_1[\partial, s]$, where $d_1 \in A_1$ and $d = d_1\partial + a_0$. By induction on the ∂-degree of d, one has $[d_1, s]\partial \in bA_1$. Since $deg_\partial(d_1) = p - 1$, it is also clear that $d_1b^p \in bA_1$. Finally $[d, s] \in bA_1$.

By lemma 9 above, one has $f_i \cdot s \in \mathcal{D}(R, V)$ and $[f_i, s] \in qA_1$ for each i.

$$s \cdot v_i = s \cdot (f_i(1)) = (s \cdot f_i)(1) = (f_i \cdot s + [f_i, s])(1)$$

One has $(f \cdot s)(1) \in V$, $[f_i, s](1) \in qk[t]$, it follows that $s \cdot v_i \in V$ and that ends the proof of theorem 8.

Next lemma justify the definition we gave for primary decomposable subspaces.

Lemma 10: Let k be a field of characteristic zero and suppose there exist q an irreducible element of $k[t]$ with $deg(q) \geq 2$. If $V = k + qk[t]$, then $\mathcal{D}(R, V) = qA_1$. In particular V is not primary decomposable subspace.

Proof : Since q is irreducible, one shows by a straightforward calculation that the right ideal qA_1 is maximal. Clearly one has $qA_1 \subseteq \mathcal{D}(R, V)$, and $\mathcal{D}(R, V) \neq A_1$ since $1 \notin \mathcal{D}(R, V)$. So one has $qA_1 = \mathcal{D}(R, V)$.

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